The Hardy Operator

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Abstract

The Hardy operator is defined on sequence and function spaces. Its boundedness can be deduced from the so called Hardy inequaltity in both its discrete and continuous forms. We presented different proofs of this inequality that go back to Hardy, Elliot and Ingham among others. In [BHS65] Brown, Halmos and Shields published further aspects on the discrete unweighted Hardy operator h^2 on the Hilbert space l^2. They proved important statements about the spectra and point spectra of h^2 and its dual operator. We investigated how far the results in [BHS65] are applicable to the Hardy operator on the weighted sequence space $l^2(\mathbb{N}, \lambda)$.

Definition of the Hardy Operator

Let $p > 1$, $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ a sequence of positive weights. Let $l^p(\mathbb{N}, \lambda)$ be the space of all sequences $a = (a_n)_{n\in\mathbb{N}}$ of complex numbers with

$$\|(a_n)_{n\in\mathbb{N}}\|^p_{l^p(\mathbb{N},\lambda)} := \sum_{n=1}^{\infty} \lambda_n |a_n|^p < \infty,$$

Let L^p be the space of all (equivalence classes modolo equality almost everywhere of) measurable functions $f : (0, \infty) \to \mathbb{C}$ with

$$\|f\|^p_{L^p} := \int_0^{\infty} |f(x)|^p \, dx < \infty.$$

Let $(a_n)_{n\in\mathbb{N}} \in l^p(\mathbb{N}, \lambda)$ and $f \in L^p$.
The discrete weighted Hardy operator $h_p^{(\lambda)}$ on $l^p(\mathbb{N}, \lambda)$ is defined by

$$h_p^{(\lambda)}(a_n)_{n\in\mathbb{N}} := \left(\frac{\sum_{k=1}^n \lambda_k a_k}{\sum_{k=1}^n \lambda_k}\right)_{n\in\mathbb{N}}.$$

The continuous Hardy operator H_p on L^p is defined by

$$H_p f(x) := \frac{1}{x}\int_0^x f(t)dt$$

for (almost all) $x \in (0, \infty)$.

The Hardy Inequality

Theorem 1: The discrete weighted Hardy inequality:
Let $p > 1$, $(a_n)_{n\in\mathbb{N}} \in l^p(\mathbb{N}, \lambda)$, $a_n \geq 0$ for all $n \in \mathbb{N}$, let $\Lambda_n = \sum_{k=1}^n \lambda_k$ and $A_n = \sum_{k=1}^n \lambda_k a_k$. Then:

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p.$$

Sketch of proof (after Elliot):

- For all $n \in \mathbb{N}$ we calculate $\left(\frac{A_n}{\Lambda_n}\right)^p - \frac{p}{p-1} a_n \left(\frac{A_n}{\Lambda_n}\right)^{p-1}$ and obtain for arbitrary $N \in \mathbb{N}$:

$$\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p - \frac{p}{p-1}\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{p-1} a_n \leq 0.$$

- Applying the Hölder inequality completes the proof.

□

Theorem 2: The continuous Hardy inequality:
Let $p > 1$, $f \in L^p$, $f \geq 0$. Let $F(x) = \int_0^x f(t)dt$ for all $x \in (0, \infty)$, then:

$$\int_0^{\infty} \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p dx.$$

Sketch of proof (after Ingham):

$$\left\|\frac{1}{x}\int_0^x f(t)dt\right\|_{L^p} = \left\|\int_0^1 f(sx)ds\right\|_{L^p} \leq \int_0^1 \|f(sx)\|_{L^p}\, ds$$
$$= \int_0^1 \left(\int_0^{\infty} f(sx)^p dx\right)^{\frac{1}{p}} ds = \int_0^1 \left(\int_0^{\infty} f(t)^p \frac{dt}{s}\right)^{\frac{1}{p}} ds = \frac{p}{p-1}\|f\|_{L^p}.$$

□

The work of Brown, Halmos and Shields

Properties of the Hardy operator h_2 on $l^2 (= l^2(\mathbb{N}, (1)_{n\in\mathbb{N}}))$:

- The dual operator h_2^* of h_2 on l^2 is defined by

$$h_2^* : l^2 \to l^2,\ (a_n)_{n\in\mathbb{N}} \mapsto \left(\sum_{k=n}^{\infty} \frac{a_k}{k}\right)_{n\in\mathbb{N}}.$$

- Let $d_2 : l^2 \to l^2$, $(a_n)_{n\in\mathbb{N}} \longmapsto \left(\frac{a_n}{n}\right)_{n\in\mathbb{N}}$ and id be the identic map on l^2, then:

$$(id - h_2)(id - h_2^*) = (id - d_2) \text{ and } \|id - h_2\|_{op} = 1.$$

- For all $0 \neq a \in l^2$: $\|(id - h_2)a\|_{l^2} < \|a\|_{l^2}$.

Theorem 3: The spectra of h_2 and h_2^*:

1. The point spectrum of h_2 is **empty**.
2. If $|1 - \lambda| < 1$, then λ is a simple eigenvalue of h_2^*.
3. The point spectrum of h_2^* is the **open disc** $\{\lambda : |1 - \lambda| < 1\}$.
4. The spectrum of h_2 is the **closed disc** $\{\lambda : |1 - \lambda| \leq 1\}$.

Results for the weighted Hardy operator on $l^2(\mathbb{N}, \lambda)$

As in [BHS65] we deduced:

- The dual operator $h_2^{(\lambda)*}$ of $h_2^{(\lambda)}$ on $l^2(\mathbb{N}, \lambda)$ is defined by

$$h_2^{(\lambda)*} : l^2(\mathbb{N}, \lambda) \to l^2(\mathbb{N}, \lambda),\ (a_n)_{n\in\mathbb{N}} \mapsto \left(\sum_{k=n}^{\infty} \frac{\lambda_k a_k}{\Lambda_k}\right)_{n\in\mathbb{N}}.$$

- Let $d_2^{(\lambda)} : l^2(\mathbb{N}, \lambda) \to l^2(\mathbb{N}, \lambda)$, $(a_n)_{n\in\mathbb{N}} \longmapsto \left(\frac{\lambda_n a_n}{\Lambda_n}\right)_{n\in\mathbb{N}}$ and id be the identic map on $l^2(\mathbb{N}, \lambda)$, then:

$$(id - h_2^{(\lambda)})(id - h_2^{(\lambda)*}) = (id - d_2^{(\lambda)}) \text{ and } \left\|id - h_2^{(\lambda)}\right\|_{op} \leq 1.$$

- For all $0 \neq a \in l^2(\mathbb{N}, \lambda)$: $\left\|(id - h_2^{(\lambda)})a\right\| < \|a\|$.

For the spectra of $h_2^{(\lambda)}$ and its dual we could infer:

Theorem 4:

1. The point spectra of $h_2^{(\lambda)}$ and $h_2^{(\lambda)*}$ are included in the open disc $\{\mu \in \mathbb{C} : |1 - \mu| < 1\}$.
2. The spectra of $h_2^{(\lambda)}$ and $h_2^{(\lambda)*}$ are included in the closed disc $\{\mu \in \mathbb{C} : |1 - \mu| \leq 1\}$.

Theorem 5:
The point spectrum of $h_2^{(\lambda)}$ in general is **not empty**.

Theorem 6:
The point spectrum of $h_2^{(\lambda)*}$ in general is not the complete open disc $\{\mu : |1 - \mu| < 1\}$.

References

[BHS65] A. Brown, P. R. Halmos and A. L. Shields. Cesaro operàtors. *Acta Sci. Math. (Szeged)* 26:125-137, 1965.

[Cop27] E. T. Copson. Note on a series of positive terms. *J. London Math. Soc.* 2:9-12, 1927.

[Har20] G. H. Hardy. Notes on a Theorem of Hilbert. *Math. Z.* 6:314-317, 1920.

[Har25] G. H. Hardy. Notes on some points in the integral calculus, LX. An inequality between integrals. *Messenger of Math.* 54:150-156, 1925.

[HLP78] G. H. Hardy, J. E. Littlewood and G. Pólya. *Inequalities*, 2nd ed.. Cambridge University Press, Cambridge, 1978.

[KMP06] A. Kufner, L. Maligranda and L.-E. Persson. The prehistory of the Hardy inequality. *American Math.Month.*, pp. 715-732, October 2006.

[Lan26] E. Landau. A note on a theorem concerning series of positive terms: Extract from a letter of Prof. E. Landau to Prof. I. Schur. *J. London Math. Soc.* 1:38-39, 1926.