On the chromatic number of random graphs and a modified version of the chromatic number

Markus Reiter, Prof. Dr. Anusch Taraz
Center for Mathematical Sciences, Technische Universität München
Chair of Applied Geometry and Discrete Mathematics

Abstract

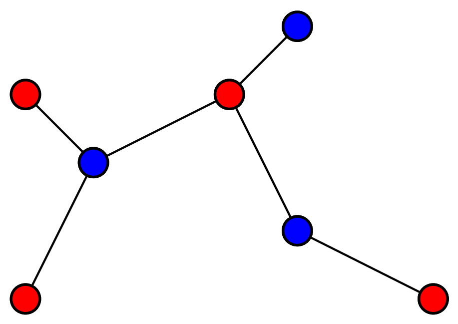
In this thesis a modified version of the chromatic number is defined and it's asymptotic behaviour is analysed. Using bounds on the usual chromatic number, the asymptotic behaviour of this modified version can be determined for a considerable range of probabilities. Moreover Łuczak's proof for the asymptotic behaviour of the chromatic number is analysed and explained in detail.

Definition

Def: **Chromatic Number**:
A graph $G = (V, E)$ is k-colourable, if there exists a map $f : V \to [k]$ with

$$f(v) \neq f(w) \text{ for every } \{v, w\} \in E$$

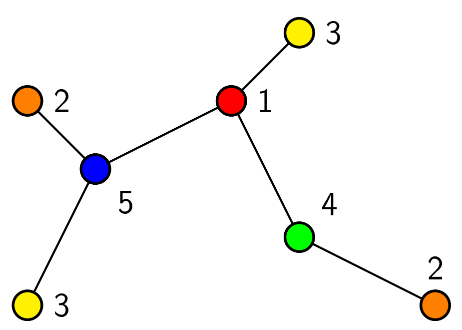
The smallest natural number k, for which G is k-colourable, is called the chromatic number $\chi(G)$.



Def: **$L_{2,1}$-chromatic number**:
A graph $G = (V, E)$ is $L_{2,1}$-colourable with k colours, if there exists a map $f : V \to [k]$ satisfying the following two properties

$$|f(v) - f(w)| \geq 2 \text{ for every } \{v, w\} \in E$$
$$|f(v) - f(w)| \geq 1 \text{ for every } v,\ w \text{ which can be connected by a path of length 2}$$

The smallest natural number k, for which G is $L_{2,1}$-colourable with k colours, is called the $L_{2,1}$-chromatic number $\chi_{2,1}(G)$.



Results

Lemma 1: **Graphs with diameter 2**:
Let $G(n, p)$ be a random graph with edge probability $p = p(n)$. If

$$p^2 n - 2 \log n \to \infty \text{ for } n \to \infty$$

then $\operatorname{diam} G \leq 2$ asymptotically almost surely.

Corollary: **A lower bound**:
Let $G(n, p)$ be a random graph with edge probability

$$p = p(n) \geq \sqrt{\frac{(2 + \varepsilon) \log n}{n}}$$

for some $\varepsilon > 0$. Then $\chi_{2,1}(G(n, p)) \geq n$ asymptotically almost surely.

Lemma 2: **Upper bound dependent on the chromatic number**:
For any graph G the following holds:

$$\chi_{2,1}(G) \leq \chi(G) + n - 1$$

Theorem: **Asymptotic behaviour of $L_{2,1}$-colouring**:
Let $0 < \varepsilon < 1$. Assume that

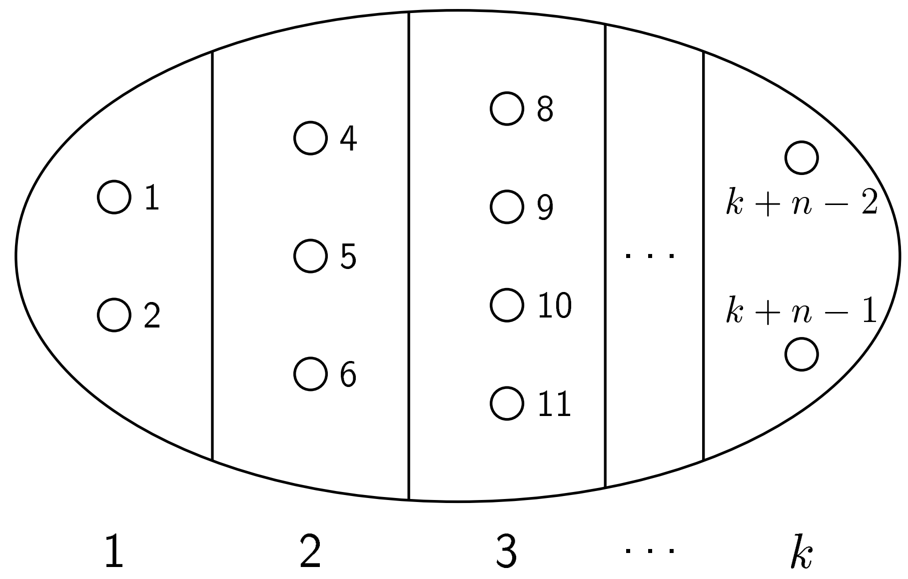
$$\sqrt{\frac{(2 + \varepsilon) \log n}{n}} \leq p(n) \leq 1 - \varepsilon$$

for large n. Then asymptotically almost surely the following is true:

$$n \leq \chi_{2,1}(G) \leq n \left(1 + \frac{2}{3 \log n}\right)$$

Proof sketch for Lemma 2

Let $f : V \to [k]$ be a k-colouring. Without loss of generality $f(v) \leq f(w)$ for $v < w$. Define $g : V \to [k + n - 1]$ by setting $g(v) = f(v) + v - 1$.



Then g is a $L_{2,1}$-colouring with $k + n - 1$ colours. Using an optimal colouring f we obtain

$$\chi_{2,1}(G) \leq \chi(G) + n - 1$$

History of bounds on the chromatic number

Grimmet, McDiarmid, 1975:
Let $0 < p < 1$ and $\varepsilon > 0$. Then asymptotically almost surely

$$\left(\frac{1}{2} - \varepsilon\right) \log \frac{1}{1-p} \cdot \frac{n}{\log n} \leq \chi(G(n,p)) \leq (1 + \varepsilon) \log \frac{1}{1-p} \cdot \frac{n}{\log n}$$

Matula, 1987:
Let $0 < p < 1$ and $\varepsilon > 0$. Then asymptotically almost surely

$$\left(\frac{1}{2} - \varepsilon\right) \log \frac{1}{1-p} \cdot \frac{n}{\log n} \leq \chi(G(n,p)) \leq \left(\frac{2}{3} + \varepsilon\right) \log \frac{1}{1-p} \cdot \frac{n}{\log n}$$

Bollobás, 1988:
Let $0 < p < 1$ be fixed and set $q = 1 - p$, $d = 1/q$ and

$$s_0 = \lceil 2 \log_d n - \log_d \log_d n + 2 \log_d (e/2) + 1 \rceil$$

Then almost every $G(n, p)$ is such that

$$\frac{n}{s_0} \leq \chi(G(n,p)) \leq \frac{n}{s_0} \left(1 + \frac{3 \log \log n}{\log n}\right)$$

Bollobás, 1988:
Let $0 < \theta < 1/3$ be fixed, let $p = n^{-\theta}$ and $s = \lfloor n^{\theta} \{2(1 - \theta) \log n - 2 \log \log n\} \rfloor$. Then almost every $G(n, p)$ is such that

$$\frac{n}{s} \leq \chi(G(n,p)) \leq \frac{n}{s} \left(1 + \frac{4 \log \log n}{(1 - \theta) \log n}\right)$$

Łuczak, 1991:
There exists a constant d_0 such that if $d = d(n) = np(n) > d_0$ and $p(n) \to 0$ then asymptotically almost surely

$$\frac{d}{2 \log d} \left(1 + \frac{\log \log d - 1}{\log d}\right) < \chi(G(n,p)) < \frac{d}{2 \log d} \left(1 + \frac{30 \log \log d}{\log d}\right)$$

References

1. Grimmet, G. R., and McDiarmid, C. On colouring random graphs. *Math. Proc. Camb. Phil. Soc. 77* (1974), 313-324
2. Matula, D. W. Expose-and-merge exploration and the chromatic number of a random graph. *Combinatorica 7*, 3 (1987), 275-284
3. Bollobás, B. The chromatic number of random graphs. *Combinatorica 8*, 1 (1988), 49-55
4. Luczak, T. The chromatic number of random graphs. *Combinatorica 11*, 1 (1991), 45-54