Finite-Element Discretization for a Model Problem in Shape Optimization

Bernhard Kiniger, Prof. Dr. Boris Vexler
Center for Mathematical Sciences, Technische Universität München
Lehrstuhl für Mathematische Optimierung

Abstract

Shape optimization problems are optimal control problems where the control is the domain, the state is the solution of a partial differential equation on that domain. We consider a model problem, where the variable part of the boundary of the domain is described as a graph of a sufficiently regular function. First of all, the existence of at least one globally optimal solution is being shown. In order to solve the problem numerically, it is being discretized using piecewise linear finite elements for the state and piecewise polynomials for the control. A priori error estimates for the control error are also derived and checked numerically. The implementation is done in C^{++}.

The Problem

We consider the following problem

$$\mathbb{P} \qquad \min_{q\in Q^{ad},\, u\in H_0^1(\Omega_q)} J(q,u) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega_q)} + \frac{\alpha}{2}\|q''\|^2_{L^2((0,1))}$$

subject to

$$(1) \qquad \begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega_q \\ u &= 0 \quad \text{on } \Gamma_q = \partial\Omega_q \end{aligned}$$

$$\Omega_q = \{(x,y)\in\mathbb{R}\times\mathbb{R} \,|\, x\in(0,1),\ y\in(q(x),1)\}$$
$$Q^{ad} = \left\{q\in H_0^1((0,1))\cap H^2((0,1)) \,\middle|\, q\le 1-\overline{\varepsilon} \text{ for a given } \overline{\varepsilon}>0\right\}$$

i.e. we search for a *control* $q\in Q^{ad}$ such that the corresponding *state* $u=u(q)$, defined as the weak solution of (1), is as close as possible to a given desired state $u_d\in C^{2,1}(\Omega_q)$. The constant $\alpha>0$ is a parameter that regularizes the solution and even ensures that $\mathbb{P}$ has a solution. The existence of an *optimal solution* $(\overline{q},\overline{u})$ can be shown by standard arguments. Although the functional J is convex, the assignment $q\to u(q)$ is nonlinear, so the *reduced cost functional* $j(q)=J(q,u(q))$ need not be convex, hence the optimal solution $(\overline{q},\overline{u})$ need not be unique.

Discretization

In order to solve $\mathbb{P}$ numerically one has to discretize the control as well as the state.

Discretization of the Control

We discretize the control as piecewise polynomials of degree at most 3, so

$$q_\sigma\in Q^{ad}_\sigma = \left\{q_\sigma\in Q^{ad} \,\middle|\, q_\sigma|_{I_i}\in\mathcal{P}_3(I_i)\,\forall i\in\{1,\dots,N\}\right\}$$

for a given *subdivision* of $[0,1]=\overline{I}_1\cup\ldots\cup\overline{I}_N$ with $\sigma=\max_i|I_i|$ being a discretization parameter. The *partially discretized problem* now reads as

$$\mathbb{P}_\sigma \qquad \min_{q_\sigma\in Q^{ad}_\sigma} j(q_\sigma)$$

Discretization of the State

We first choose a *reference domain* $\Omega_0=(0,1)^2$ and just discretize the *transported solution* $u\circ T_q\in H_0^1(\Omega_0)$. Take a family of *triangulations* $(\mathcal{T}_h)_{h>0}$ of Ω_0 satisfying the "usual" regularity assumptions and define the space of the discrete test functions as

$$V_h = \left\{v^h\in H_0^1(\Omega_0) \,\middle|\, v^h|_K\in\mathcal{P}_1(K)\,\forall K\in\mathcal{T}_h\right\}$$

The *fully discretized problem* now reads as

$$\mathbb{P}_{\sigma,h} \qquad \min_{q_\sigma\in Q^{ad}_\sigma,\, u^h\in V_h} j_h(q_\sigma) = J\left(q_\sigma, u^h\circ T^{-1}_{q_\sigma}\right)$$

subject to

$$(2) \qquad \int_{\Omega_0}\nabla u^h A_{q_\sigma}\nabla z^h + u^h z^h\gamma_{q_\sigma}\,\mathrm{d}x = \int_{\Omega_0} f\circ T_{q_\sigma} z^h\gamma_{q_\sigma}\,\mathrm{d}x \quad \forall z^h\in V_h$$

where the coefficients of A_{q_σ} depend on q_σ in a nonlinear way. One should also remark that (2) is just a transportation of (1) onto the domain Ω_0.
It is possible to show that $\mathbb{P}_\sigma$ as well as $\mathbb{P}_{\sigma,h}$ have at least one *globally* optimal solution, but they can be far from any solution of $\mathbb{P}$. However, under some assumptions it can be shown that for each optimal control $\overline{q}$ of $\mathbb{P}$ there exists a sequence $(\overline{q}_{\sigma,h})_{\sigma,h>0}$ of *local* optimal solutions of $\mathbb{P}_{\sigma,h}$ with

$$(3) \qquad \|\overline{q}-\overline{q}_{\sigma,h}\|_{H^2((0,1))}\to 0 \quad \text{for } \sigma,h\to 0$$

Computation of the Derivative j'

Lemma 1. *The derivative of $j(q)$ is given as*

$$j'(q)(\delta q) = \int_{\Gamma_q}\left(\frac{1}{2}(u-u_d)^2 + \partial_n u\cdot\partial_n z\right)\cdot\langle V_{q,\delta q}, n\rangle\,\mathrm{d}x + \alpha\int_0^1 q''\cdot\delta q''\,\mathrm{d}x$$

where the adjoint state $z\in H_0^1(\Omega_q)$ solves the adjoint equation

$$-\Delta z + z = u-u_d \text{ in } \Omega_q,\ z=0 \text{ on } \Gamma_q$$

and the vector field $V_{q,\delta q} = \left(0,(1-y)\frac{\delta q(x)}{1-q(x)}\right)^T$ defines a transformation from Ω_q to $\Omega_{q+\delta q}$.

By exploiting the *first-order optimality conditions*

$$j'(\overline{q})(\delta q) = 0 \quad \forall\delta q\in C_0^\infty((0,1))$$

(we assume that the constraint $\overline{q}\le 1-\overline{\varepsilon}$ is nonactive) one can show

Theorem 2. *For any optimal solution of $\mathbb{P}$ it holds the improved regularity*

$$(\overline{q},\overline{u})\in H^4((0,1))\times H^2(\Omega_{\overline{q}})$$

A Priori Error Estimates and Numerical Results

For a sequence as given in (3) one can show that

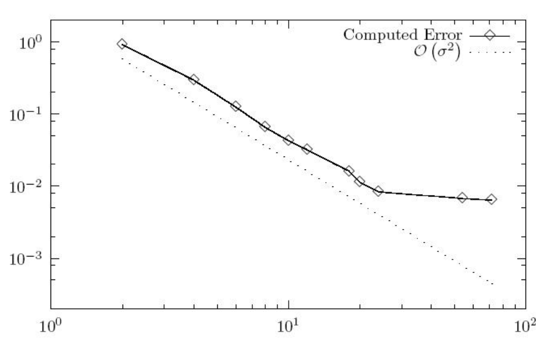
Theorem 3. *There exist $c_1,c_2>0$ independent of σ,h with*

$$\|\overline{q}-\overline{q}_{\sigma,h}\|_{H^2((0,1))}\le c_1\cdot\sigma^2 + c_2\cdot h$$

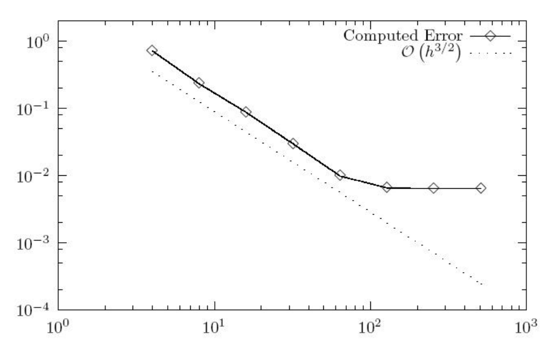
The problem has been implemented in the C^{++}-Libraries RoDoBo/Gascoigne, where it has been solved by applying a Newton method to the reduced problem $\mathbb{P}_{\sigma,h}$. The following table shows that the number of iterations is basically mesh-independent

σ \ h	16^{-1}	32^{-1}	64^{-1}	128^{-1}	256^{-1}
10^{-1}	3	3	3	3	3
20^{-1}	3	6	4	4	3
30^{-1}	-	4	4	4	5

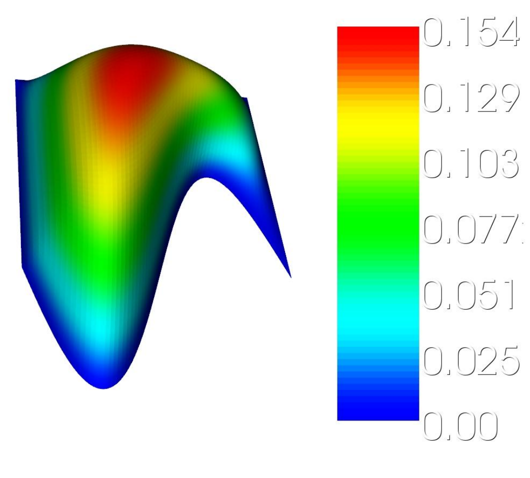
For σ we observed, as predicted, an error of $\mathcal{O}(\sigma^2)$, whereas for the error in h we even observed a higher convergence of order $h^{3/2}$.



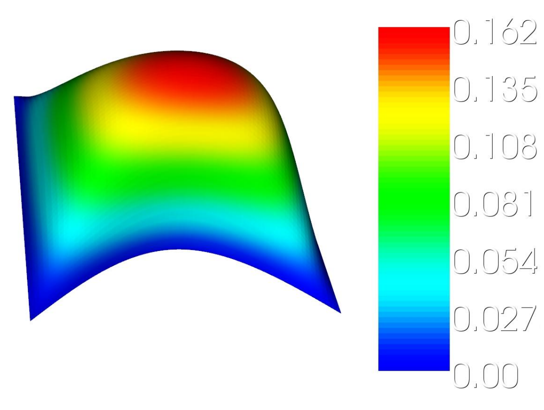
Error in σ



Error in h



Initial state



Optimal state