Discrete to continuum Γ-convergence in brittle materials: A two-dimensional crack model

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Abstract

Fracture Mechanics is a field in solid mechanics dealing with the behaviour of brittle bodies under deformation, in particular with the initiation and formation of cracks. In a variational description the deformation of brittle materials is associated with a certain energy functional and following the principle of least energy one wants to find the minimizers under some given boundary data. A great variety of one-dimensional models was discussed but physically reasonable more dimensional problems are subject to active research. The scope of the thesis is to consider a two-dimenisonal rectangular body with a triangular lattice and to investigate the behaviour of the deformation under tensile boundary conditions. On this poster we first discuss typical energy functionals in Fracture Mechanics, then verify that they can be seen as suitable limits of discrete systems via Γ-convergence. Finally, we deal with our two-dimensional crack model.

Energy functionals and minimizing problems

One dimension

The energy of a one-dimensional bar of length l with (longitudinal) displacement $u : (0,l) \to \mathbb{R}$ is given by

$$E(u) = \int_0^l F(\dot{u}) + \sum_{t \in S(u)} G([u](t)).$$

Typically we have two contributions:

- Bulk term: F convex vanishing at 0
- Surface term:
  1. Sum over discontinuity points $S(u)$ with jump height $[u](t) = u(t+) - u(t-)$
  2. $G = \infty$ on $(-\infty, 0)$ (concerning impenetrability)
  3. G increasing and concave on $(0,\infty)$, $\lim_{x\to\infty} G(x) = \beta$.

We can even consider an easier surface term assuming $G = \beta$ on $(0,\infty)$ (Griffith energy). To understand the behaviour of the body we want to solve

$$\min\left\{E(u) \middle| \; u \in P - W^{1,p}, \quad u(0) = 0, \; u(l) = d \right\}.$$

For $d > 0$ the bar is stretched, $d < 0$ is the case of compression. The admissible functions are the piecewise weakly differentiable functions: $P - W^{1,p} = W^{1,p} + PC$, where PC denotes piecewise constant functions. The exponent p concerns a growth condition for the bulk density F.

More dimensions

For $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}^N$ we consider the energy

$$\mathcal{E}(u) = \int_\Omega \mathcal{F}(\nabla u)\, d\mathcal{L}^N + \int_{S(u)} \mathcal{G}\left(u^+ - u^-, \nu_u\right) d\mathcal{H}^{N-1}.$$

The suitable functions are SBV^p-functions (*special functions of bounded variation*), i.e.

- $\mathcal{H}^{N-1}\left(S(u) \setminus \bigcup_{i\in\mathbb{N}} C_i\right) = 0$, C_i some C^1-manifold, ν_u corresponding outer normal
- $\nabla u|_{\Omega \setminus S(u)} \in L^p\left(\Omega \setminus S(u)\right)$
- Derivative can be seen as a measure: $Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}|_{S(u)}$

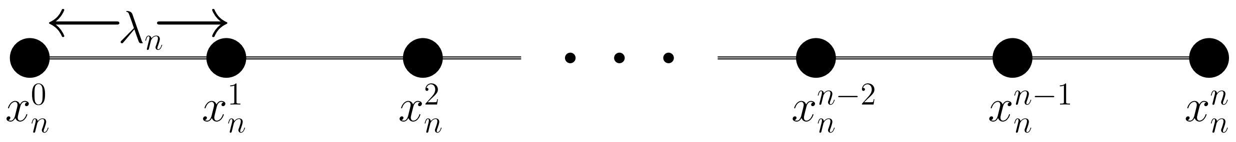
More generally the derivative could also have a diffuse Cantor part. In this case we are in the space of BV-functions. Analogously we consider the boundary value problem:

$$\min\left\{\mathcal{E}(u) \middle| \; u \in SBV^p(\Omega), \quad u = g \text{ auf } \partial\Omega\right\},$$

where $g \in SBV^p_{loc}\left(\mathbb{R}^N\right)$.

Discrete systems and Γ-convergence: One dimension

It is a natural approach to regard materials not as continua but as huge systems of atoms influencing each other. In the **one-dimensional case** we have an atom chain with $n+1$ atoms and distance $\lambda_n \to 0$ for $n \to \infty$.



Denoting u_n^i as the displacement of each atom and $\dot{u}_n^i = \frac{u_n^{i+1} - u_n^i}{\lambda_n}$ as the relative displacement the energy of the system can be written in the following way:

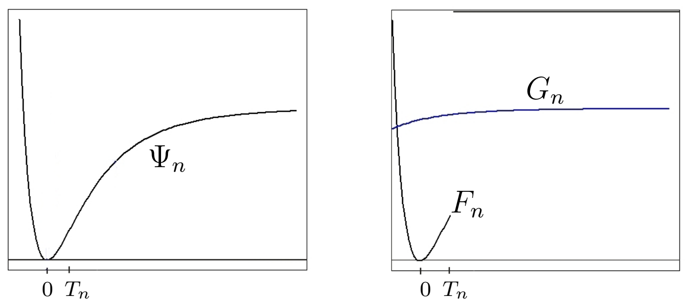
$$E_n\left(\{u_n^i\}\right) = \sum_{i=0}^n \lambda_n \Psi_n\left(\dot{u}_n^i\right).$$

Here Ψ_n is of Lennard-Jones type with $\Psi_n(0) = 0$ and convex-concave passage at T_n. The following result (cf. [1],[3]) provides a connection between these discrete functionals and the continuous energy functional.

Theorem 1. *Under suitable growth conditions for Ψ_n and $T_n \to \infty$ we get: $E = \Gamma - \lim E_n$ in L^1.*

Γ-convergence is a convergence for functionals which is appropriate for minimizing problems. First, one has to show that the limit energy is a lower bound. Then one proves that this lower bound is sharp by constructing a 'recovery sequence'. The crucial point here is to split up the Lennard-Jones potential in two parts F_n, G_n which convergence pointwise to the densities F,G in the continuous functional:

$$F_n(x) = \begin{cases} \Psi_n(x) & x \le T_n \\ \infty & x > T_n \end{cases}, \quad G_n(x) = \begin{cases} \lambda_n \Psi_n\left(\frac{1}{\lambda_n}x + T_n\right) & x > 0 \\ \infty & x < 0, \end{cases}$$

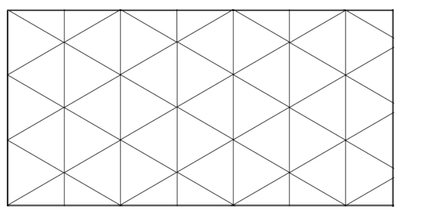


G_n is rescaled in an appropriate way since it should be only active at jump points and not on the whole interval.
Via Γ-convergence we get:

Theorem 2. *E attains its minimum under given boundary conditions. Moreover, if $E_n(u_n) = \min E_n$ for all $n \in \mathbb{N}$ and $u_n \to u$ in L^1 then $E(u) = \min E$.*

The two-dimensional crack model

The discrete-to-continuum approach cannot be simply generalized to more dimensions, only in the case of scalar valued deformation (see [2]). But this case is physically not meaningful. The crucial point why generalization fails is that the more-dimensional Lennard-Jones potential is not convex in neighbourhoods of its global minima. Our scope is to treat a two-dimensional model with vector valued deformation.



Although we cannot find a Γ-limit on whole L^1 we calculate at least the 'right' continuous energy for the minimizers of boundary value problems with homogeneous tensile boundary conditions. We assume a two-dimensional rectangular body with a triangular lattice (lattice directions ν_1, ν_2, ν_3).

The discrete energies can be written as a sum of Lennard-Jones interactions (y_n denotes the deformation):

$$E_n(u_n) = \sum_{i=1}^3 \sum_{\substack{x,z \in \mathcal{G}_n, \\ y = x + \lambda_n \nu_i}} \frac{3}{4} \lambda_n^2 \Psi_n\left(\frac{y_n(z) - y_n(x)}{\lambda_n}\right).$$

We consider the (suitably constructed) continuous energy functional

$$E(y) = \int_\Omega f(\nabla y e_1) + \int_{S(y)} \beta |\nu_y \cdot e_1| d\mathcal{H}^1$$

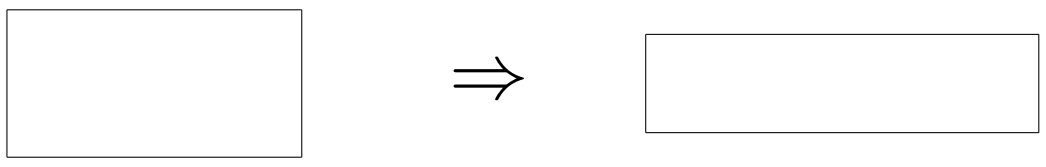
with bulk energy density $f(x) := \min_{z\in\mathbb{R}^2} F^{**}\left(\sqrt{\frac{1}{4}\|z\|^2 + \frac{3}{4}\|x\|^2}\frac{x}{\|x\|}\right) + \frac{1}{2}F(z)$.

Theorem 3. *E is a lower energy bound for the discrete energies E_n. Moreover, the bound is sharp for special minimizers of E under homogeneous tensile boundary conditions, i.e. we find a recovery sequence $y_n \to y$ s.t. $E_n(y_n) \to E(y)$.*

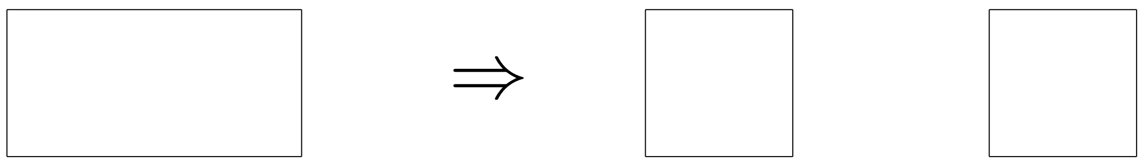
The crucial point is to establish the lower bound. The construction of the density f is the main idea in the proof. f optimizes the energy of one triangle in dependence of the derivate in e_1 direction. By a suitable relaxation we get that f is convex, i.e. we can apply lower semicontinuity results for integral functions. The 'slicing' technique is used in order to reduce the problem to one-dimensional fibers. This technique can be seen as a generalization of Fubini's theorem for functions performing jumps on subsets of codimension 1.

Results in pictures:

'Small' expansion: Expansion in e_1-, compression in e_2-direction (Possion-effect).



'Large' expansion: Even crack in e_2-direction. The two pieces are not deformed.



References

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