

Introduction

Particle sources are an indispensable part of any scattering experiment. Nevertheless in the theoretical description they are mostly disregarded on the basis that a “suitable” wave function has been prepared.

The aim of thesis [1], however, is the discussion of a simple model for a quantum particle source. There are results concerning

- the qualitative growth behaviour of the number of produced particles,
- a classical particle source as a limiting case of the model.

The Model

Consider noninteracting, spinless bosons or fermions which are generated by the source in a normalized state $\phi \in L^2(\mathbb{R}^d) =: \mathcal{H}$, $\|\phi\| = 1$.

The full dynamics of the system is given by a completely positive, quasifree semigroup on the bosonic or fermionic Fock space, and reduces to an evolution equation of the one-particle density matrix $\rho(t)$, which is a trace class operator on \mathcal{H} :

$$\frac{d}{dt}\rho = -i[H, \rho] + 2\lambda P_\phi \pm \lambda(P_\phi \rho + \rho P_\phi) \quad (t \geq 0)$$

Here, H is the Hamiltonian governing the one-particle motion, $\lambda > 0$ is a measure for the source strength, and $P_\phi := |\phi\rangle\langle\phi|$ is the projector onto the source state ϕ . The sign \pm is different for the bosonic (+) and fermionic (−) case.

The solution of the above time evolution reads as

$$\rho(t) = e^{(-iH \pm \lambda P_\phi)t}(\rho(0) \pm \text{id})e^{(iH \pm \lambda P_\phi)t} \mp \text{id}$$

The number of particles produced until time t is given by the trace $\text{tr}(\rho(t))$.

Linear growth

Theorem. For the choice of a state ϕ with an integrable overlap and small $\lambda > 0$, i.e.

$$\lambda \int_0^\infty |\langle \phi, e^{-iHt}\phi \rangle| dt < 1,$$

the number of bosons or fermions grows linear in time.

Idea of the proof: Obtain the semigroup $(e^{(-iH \pm \lambda P_\phi)t})_{t \geq 0}$

by perturbing the unitary group $(e^{-iHt})_{t \in \mathbb{R}}$.

From this, one can show that

$$\frac{d}{dt} \text{tr}(\rho_\pm(t)) = 2\lambda \|e^{(-iH \pm \lambda P_\phi)t}\phi\|^2$$

converges to a nonzero limit as $t \rightarrow \infty$.

Exponential growth

Theorem. For any $\phi \in \mathcal{H}$, there exists a $\lambda_c \geq 0$ such that for all $\lambda > \lambda_c$ the particle number grows exponentially in the bosonic case.

Idea of the proof:

- After projecting to a suitable finite-energy space of H , and for large λ , one can consider $\frac{iH}{\lambda}$ as a small perturbation of P_ϕ .
- With a second perturbation argument, it is possible to remove the energy cut-off again.
- In these two steps, the positive eigenvalue of P_ϕ carries over to an eigenvalue of the semigroup generator $iH + \lambda\phi$ with positive real part.

Semiclassical limit

To compare the quantum model to a time evolution in classical phase space, one defines a scaled Wigner transform of ρ on classical phase space $\mathbb{R}_x^d \times \mathbb{R}_p^d$ by

$$f^\epsilon(x, p, t) = (2\pi)^{-d} \int_{\mathbb{R}^d} \rho^\epsilon \left(\frac{x+y}{\epsilon}, \frac{x-y}{\epsilon}, \frac{t}{\epsilon} \right) dy$$

where $\rho(\dots, T)$ denotes kernel of $\rho^\epsilon(T)$, the Hamiltonian is the free kinetic energy $H = -\frac{\Delta}{2}$ and the source strength depends also on ϵ by $\lambda = c\epsilon$, $c > 0$.

In another context, this scaling of Wigner functions is discussed in [2].

Theorem. As $\epsilon \rightarrow 0$, $f^\epsilon(t)$ converges to a limit $f^0(t)$ in the topology of $\mathcal{D}'(\mathbb{R}_x^d \times \mathbb{R}_p^d)$. The limit solves

$$\frac{d}{dt} f^0(x, p, t) = -p \cdot \nabla_x f^0(x, p, t) + 2c\delta(x) |\hat{\phi}(p)|^2$$

in the sense of distributions on phase space.

An open problem. An important idea in the proof of the theorem is the Riemann-Lebesgue-Lemma, which applies because $H = -\frac{\Delta}{2}$ has only absolutely continuous spectrum. It ensures that the coupling term $P_\phi \rho + \rho P_\phi$ can be neglected in the limit $\epsilon \rightarrow 0$.

A semiclassical limit should also be possible for Hamiltonians with a slowly varying potential:

$$H^\epsilon = -\frac{\Delta}{2} + V(\epsilon X)$$

However, this operator might also have point spectrum, and it is not clear how to extend the above argument.

References

- [1] M. Butz, A Quantum Model for a Particle Source, *Bachelor's Thesis*, TU München, 2008
- [2] F. Nier, *Asymptotic Analysis of a Scaled Wigner Equation and Quantum Scattering*, *Transp. Theory Stat. Phys.*, **24** (4-5), p. 591–628, 1995