A Comparison of Binomial Proofs and Ceva-/Menelaus Proofs for Real Projective Incidence Theorems

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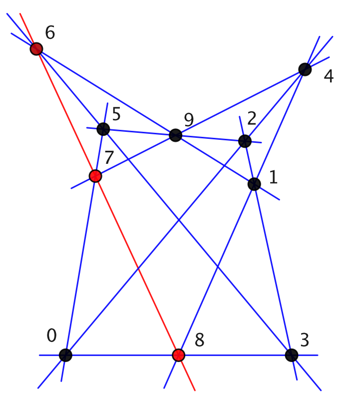
Abstract

An incidence theorem considered here takes place in $\mathbb{RP}^2$ and deals only with the question if points and lines coincide or not. The main result is that proving techniques are equivalent. One kind of proof, the binomial proof, can be interpreted as formal calculation on determinants of points using Grassmann-Plücker relations. The other proof, the Ceva/Menelaus proof, can be seen as glueing triangles together on their sides such that we end up with a closed structure. All but one triangle are equipped with certain subconfigurations corresponding to the theorems of Ceva and Menelaus. In this setting we can conclude the existence of such a subconfiguration on the last remaining triangle.

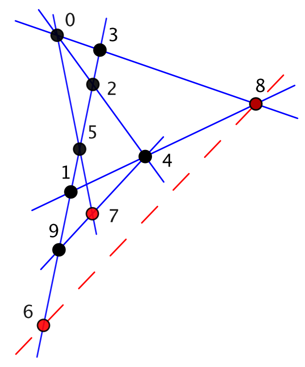
Incidence Theorems

We think of an incidence theorem as a theorem of the structure "Given a certain configuration of incidences, then also an additional incidence holds". We will see this in the concrete example of the theorem of Desargues.



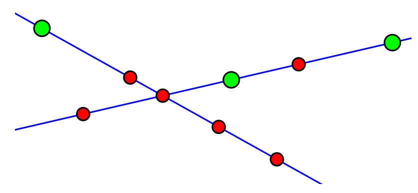
Hypotheses: Collinearities $\mathbf{H}$: $(0,3,8)$, $(0,2,4)$, $(0,5,7)$, $(1,4,8)$, $(3,5,6)$, $(1,2,3)$, $(2,5,9)$, $(1,6,9)$, $(4,7,9)$
$\Rightarrow C = (6,7,8)$!

However, in the figure below we see, that in degenerate situations, $\mathbf{H}$ is not sufficient to state this theorem:



What did happen?
A lot of lines collapsed!

We want to prevent the collapse of lines, that are related with each other. More precisely we require that lines that intersect in a point do not collapse. This can be done by requiring that some triple of points do not lie on a common line, as indicated in the next figure.



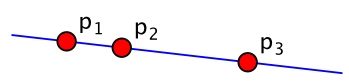
We can collect these triples of points in a set $\mathbf{B}$. So an incidence theorem is characterized by the requirements $\mathbf{H}$ and $\mathbf{B}$ and by the conclusion C.
So we write $\mathcal{T} = (\mathbf{H}, \mathbf{B}, C)$.

How can $\mathbf{H}$ and $\mathbf{B}$ be interpreted in $\mathbb{RP}^2$? The key observation is the following:
Let $p_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, p_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, p_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ be the homogenous coordinates of three points in $\mathbb{RP}^2$.

It holds: $\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = 0 \quad \Leftrightarrow \quad p_1, p_2, p_3$ lie on a common line

For convenience abbreviate the determinant by $[1,2,3]$ and call such an expression a *bracket*.

$$[1,2,3] = 0 \quad \Leftrightarrow$$



Now we are able to reformulate the characterization of $\mathcal{T}$ by terms of brackets.

The Binomial Proof

Grassmann-Plücker relations

Let $p_1, \ldots, p_5$ be in $\mathbb{RP}^2$. Then the following Grassmann-Plücker relation holds:

$$[1,2,3][1,4,5] \;-\; [1,2,4][1,3,5] \;+\; [1,2,5][1,3,4] \;=\; 0$$

So if $[1,2,3] = 0$, which is in particular the case if $(1,2,3) \in \mathbf{H}$, we have:

$$\boxed{[1,2,4][1,3,5] \;=\; [1,2,5][1,3,4] \quad \text{- a } \textit{biquadratic equation}}$$

We will see in an example for the theorem of Desargues (see figure above) how we can use these basic building blocks to obtain a proof (the leftmost column indicates triples in $\mathbf{H}$).

The Proof

$$\begin{array}{lcl}
(4,7,9) & \Longrightarrow & [4,7,1][4,9,6] = [4,7,6][4,9,1] \\
(9,1,6) & \Longrightarrow & [9,1,4][9,6,2] = [9,1,2][9,6,4] \\
(2,5,9) & \Longrightarrow & [2,5,6][2,9,1] = [2,5,1][2,9,6] \\
(2,4,0) & \Longrightarrow & [2,4,8][2,0,3] = [2,4,3][2,0,8] \\
(0,8,3) & \Longrightarrow & [0,8,2][0,3,5] = [0,8,5][0,3,2] \\
(5,7,0) & \Longrightarrow & [5,7,3][5,0,8] = [5,7,8][5,0,3] \\
(2,1,3) & \Longrightarrow & [2,1,5][2,3,4] = [2,1,4][2,3,5] \\
(4,1,8) & \Longrightarrow & [4,1,2][4,8,7] = [4,1,7][4,8,2] \\
(5,3,6) & \Longrightarrow & [5,3,2][5,6,7] = [5,3,7][5,6,2]
\end{array}$$

multiplying all equations and canceling leads to:

$$(7,6,8) \text{ or } (7,4,5) \quad \Longleftarrow \quad [7,6,4][7,8,5] = [7,6,5][7,8,4]$$

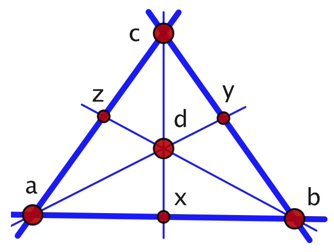
In this procedure it was essential, that we can divide by some brackets. Since we do not want to divide by zero, we require that all brackets by which one wants to divide lie in $\mathbf{B}$. The last equation gives us that the product of two determinants is zero. So if we claim (by $\mathbf{B}$) that one of those is not zero we can conclude the collinearity of three points. This describes the essential ideas of a binomial proof.

A question investigated: is it possible that the last equation evaluates to $0 = 0$? Roughly speaking the answer is: No, if the theorem is stated in a good way. A little more precisely this is the case if it is not trivial and if we can find a point configuration fulfilling the hypotheses.

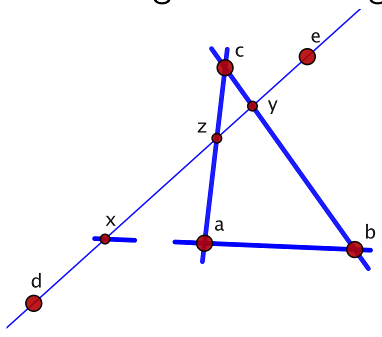
The Ceva/Menelaus Proof

The Theorems of Ceva and Menelaus

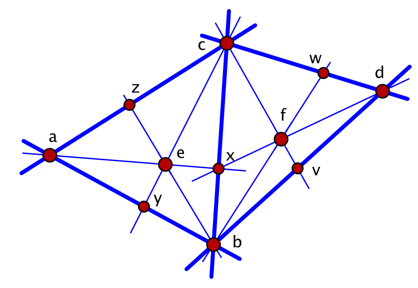
In an affine setup and considering oriented lengths, the theorems can be stated and combined as



$$\frac{|ax|}{|xb|} \cdot \frac{|by|}{|yc|} \cdot \frac{|cz|}{|za|} = 1$$



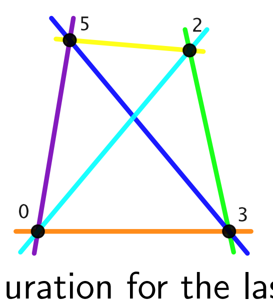
$$\frac{|ax|}{|xb|} \cdot \frac{|by|}{|yc|} \cdot \frac{|cz|}{|za|} = -1$$

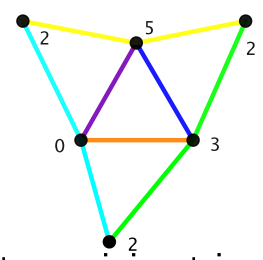


$$\frac{|ay|}{|yb|} \cdot \frac{|cz|}{|za|} \cdot \frac{|bv|}{|vd|} \cdot \frac{|dw|}{|wc|} = 1$$

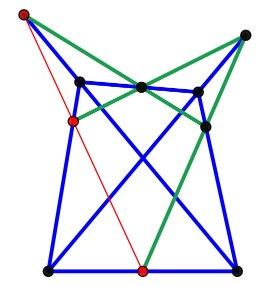
A Closed Structure of Triangles

In Desargues' theorem we can find a closed structure of four triangles as indicated in the figures below.

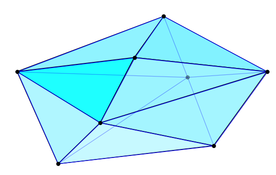




We equip all but one triangle with a Menelaus line (green in the figure below). At the edges, the triangles share all points. If we multiply the corresponding equations, we end up with the equation corresponding to a Menelaus subconfiguration for the last remaining triangle. Thus, we can equip this triangle with the given cutting points with a Menelaus line as well!

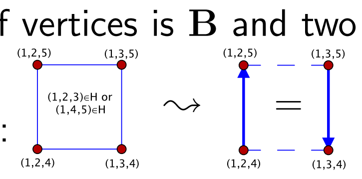


This procedure can be generalized.
Such closed structures appear naturally on triangulated manifoldes. These can be used as framework to construct new theorems.



The Base Graph Γ as Link between the Two Worlds

Consider the **base graph** Γ of a theorem $\mathcal{T}$, where the set of vertices is $\mathbf{B}$ and two vertices are connected if they differ by one element.
Here a biquadratic equation (with labeling as before) reads as:

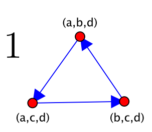


Directed edges $\big((a,b,c),(a,b,d)\big)$ in Γ can be interpreted as $\frac{[a,b,c]}{[a,b,d]}$. Furthermore an oriented length can be calculated with determinants:

$$\frac{|ax|}{|xb|} = \frac{[a,p,q]}{[x,q,p]}$$ where p and q span a line which intersects $\overline{ab}$ in x

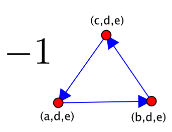
Thus the theorems of Ceva and Menelaus can be rewritten and thereby proved as follows. The situation in Γ is shown, too.

$$\frac{[a,d,c]}{[b,c,d]} \cdot \frac{[b,d,a]}{[c,a,d]} \cdot \frac{[c,d,b]}{[a,b,d]} = 1$$

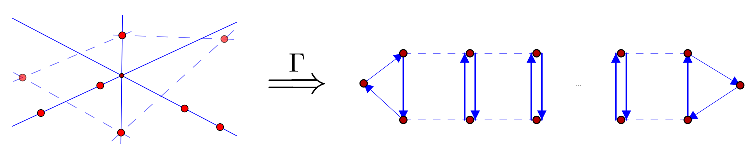


and

$$\frac{[a,d,e]}{[b,e,d]} \cdot \frac{[b,d,e]}{[c,e,d]} \cdot \frac{[c,d,e]}{[a,e,d]} = -1$$



Given a Ceva/Menelaus proof. Consider two triangles glued together as before. With some technical effort one can find a partial structure in Γ. For this one uses the fact that the triangles have common points.



We multiply all these chains in their determinantal notation. Consider a graph-triangle. There we can do the same as in the determinantal proofs just seen. This will lead us to the ends of another chain in which the conclusion C is encoded.

Given a binomial proof. To interpret it in Γ, we extract edges out of each biquadratic equation (including the final equation under the horicontal line on the left) as seen above. Multiplying all these "edges" gives 1. In the world of the base graphes this says that the edges form cycles. We consider all such cycles that have a length greater than 2. With the help of additional generic points (they do not lie on a line spanned by points of the theorem) we are able to decompose the cycles into triangles as shown here in a small example. One can show, that reinterpreting these triangles as just seen above gives a valid Ceva/Menelaus proof for $\mathcal{T}$.

References

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