Scaling behaviour of the directed polymer model of Baryshnikov and O'Connell at zero temperature

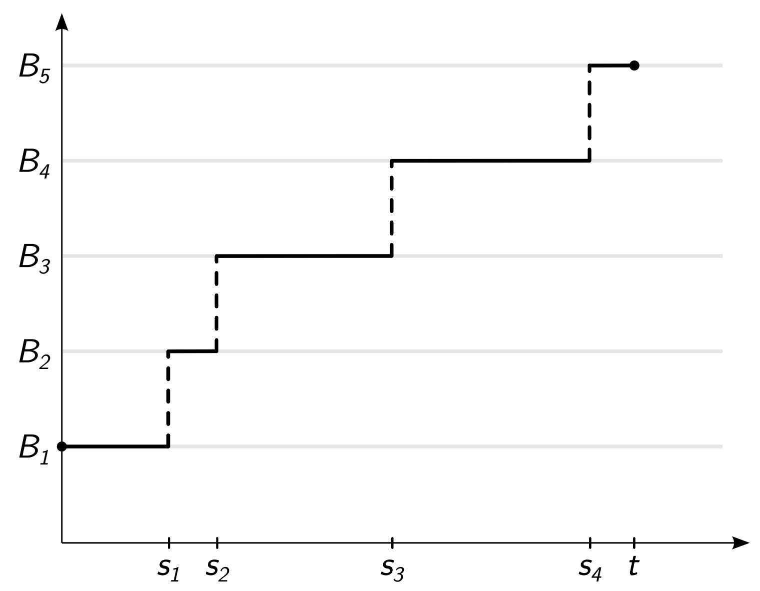
Thomas Weiß, Prof. Dr. Herbert Spohn
Zentrum Mathematik, Technische Universität München
Lehrstuhl für mathematische Physik

Abstract

The bachelor's thesis [1] studies the scaling limit of the directed polymer model of Baryshnikov and O'Connell at temperature zero. After presenting related models, a few applications and known results, some necessary facts from the theory of random matrices and determinantal point processes are introduced. The main theorem, the convergence of the random free energy as a stochastic process in time to the Airy process, is proved using the equivalence to the non-colliding Brownian motion and its representation as a determinantal process.

The model

The growth of a polymer is modeled in a $1+1$-dimensional (one time and one space dimension) random environment. Here the time t is considered continuous, while the space i is discrete. Furthermore one requires the polymer to be directed, i.e. it can grow only in positive t and i direction. The random potential of the environment is modeled as independent Gaussian white noises evolving in t at each point i of the space. The polymer then collects all the energy on its path, giving the sum of integrals of the white noises, thus brownian motions.



Viewing this as a thermodynamic system, i.e. the canonical ensemble one can define the (random) partition function at temperature $T = 1/\beta$,

$$Z_t^N(\beta) = \int\limits_{0<s_1<\cdots<s_{N-1}<t} e^{\beta(B_1(s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t))} ds_1 ds_2 \dots ds_{N-1},$$

and the corresponding free energy $\frac{1}{\beta}\log Z_t^N(\beta)$.
However, the actual object of study will be the free energy in the limit $T \to 0$ or $\beta \to \infty$ where the formula reduces to:

$$M_N(t) = \sup_{0<s_1<\cdots<s_{N-1}<t} \left[B_1(s_1) + B_2(s_1,s_2) + \cdots + B_N(s_{N-1},t)\right].$$

Dyson's Brownian motion

Let $B_{ii}(t), 1 \le i \le N$ and $B_{ij}(t), B'_{ij}(t), 1 \le i < j \le N$ be N^2 independet standard Brownian motions. Define a stochastic process $X(t)$, called *Dyson's Brownian motion*, on the space of $N \times N$ Hermitian matrices by:

$$X_{ii}(t) = B_{ii}(t)$$
$$X_{ij}(t) = X^*_{ji}(t) = \frac{1}{\sqrt{2}}\left(B_{ij}(t) + iB'_{ij}(t)\right)$$

$X(1) \stackrel{d}{=} X(t)/\sqrt{t}$ is distributed according to the *Gaussian Unitary Ensemble*; Dyson's Brownian motion is therefore a dynamical analogue of this random matrix.
Consider the process $\boldsymbol{\lambda}(t) = (\lambda_1(t), \dots, \lambda_N(t))$. With probability one the lines λ_i do not collide, i.e. $\boldsymbol{\lambda}(t)$ stays in the so called Weyl chamber of type A_{N-1}, which is $W_N = \left\{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\right\}$. Furthermore this process is equivalent to the process Ξ_N of N independent Brownian motion *conditioned* on staying in W_N at all times, called *non-colliding Brownian motion*.
The free energy of the polymer can be studied using Dyson's Brownian motion by the following theorem [2]:

Theorem 1. *The processes $(\Xi_N(t))_N$, i.e. the N-th component or top line of $\Xi_N(t)$, and $M_N(t)$ are equal in law.*

O'Connell proves an even stronger theorem involving all components of $\Xi_N(t)$. He defines a function $\Gamma_N : D_0(\mathbb{R}_+)^N \to D_0(\mathbb{R}_+)^N$, where $D_0(\mathbb{R}_+)$ denotes the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$. With a N-dimensional Brownian motion $B^{(N)}(t)$ the following identity holds:

$$\Xi_N(t) \stackrel{d}{=} \Gamma_N\left(B^{(N)}(t)\right).$$

The Result

The point of interest is the asymptotic behaviour of $M_N(t)$ as $t \to \infty, N \to \infty$ while $t/N =: \gamma$ is fixed. Since $M_N(\gamma N) \stackrel{d}{=} \sqrt{\gamma} M_N(N)$ we restrict our study to $\gamma = 1$. Previous results include an almost sure limit,

$$\frac{M_N(N)}{N} \xrightarrow[N\to\infty]{a.s.} 2,$$

and a weak limit of the asymptotic distribution,

$$\frac{M_N(N) - 2N}{N^{1/3}} \xrightarrow[N\to\infty]{d} \mathcal{T},$$

where $\mathcal{T}$ is *Tracy-Widom*-distributed.
The bachelor's thesis generalizes this last formula to a stochastic process in t. To get non-trivial results one has to allow fluctuations in time to be of order $N^{2/3}$, i.e. $2N^{2/3}\tau$, while modifying the space variable $2N$ by $2N^{2/3}\tau$, too. The following theorem gives the scaling limit of this process:

Theorem 2.

$$N^{-1/3}\left[M_N\left(N + 2N^{2/3}\tau\right) - \left(2N + 2N^{2/3}\tau\right)\right] \xrightarrow[N\to\infty]{d} \mathcal{A}(\tau) - \tau^2,$$

where $\xrightarrow{d}$ means convergence of the finite dimensional distributions.

$\mathcal{A}(t)$ is the *Airy process*, introduced by Praehofer and Spohn in [3]. It is stationary and its one-point distribution is the Tracy-Widom distribution. There is a continuous version of $\mathcal{A}(t)$, but it is not a Markov process.

The proof

The proof uses the representation of Dyson's Brownian motion as an *extended determinantal point process*. An extended point process is, roughly speaking, a random point configuration evolving in time. It is usually described by means of the $(N_1, N_2, \dots, N_M)$-point correlation function:

$$\rho_N\left(t_1, \boldsymbol{x}^{(1)}_{N_1}; t_2, \boldsymbol{x}^{(2)}_{N_2}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}\right) =$$
$$= \int\limits_{\bigotimes_{m=1}^M \mathbb{R}^{N-N_m}} p_N\left(t_1, \boldsymbol{x}^{(1)}; \dots; t_M, \boldsymbol{x}^{(M)}\right) \prod_{m=1}^M \frac{1}{(N-N_m)!} \prod_{j=N_m+1}^N dx_j^m,$$

where $\boldsymbol{x}^{(m)}_{N'} = \left(x_1^{(m)}, x_2^{(m)}, \dots, x_{N'}^{(m)}\right)$. The occuring point processes are determinantal, i.e. the correlation functions are given by the determinant of a kernel. Dyson's Brownian motion is governed by the *extended Hermite kernel*, which is more or less a sum of Hermite functions, while the Airy process is governed by the *extended Airy kernel*, which is an integral over Airy functions. With a given kernel $\mathcal{K}$, the finite-dimensional distributions can be calculated by a *Fredholm determinant*:

$$\mathbb{P}\left(x^{(i)}_{max} \le s_i,\ 1 \le i \le M\right) = \det(\mathbb{1} - \mathcal{K})_{L^2\left(\bigotimes_{i=1}^M (s_i,\infty)\right)}$$
$$= \sum_{N_1=0}^\infty \cdots \sum_{N_M=0}^\infty \prod_{m=1}^M \frac{(-1)^{N_m}}{N_m!} \int_{(s_1,\infty)^{N_1}} \prod_{j=1}^{N_1} dx_j^{(1)} \cdots \int_{(s_M,\infty)^{N_M}} \prod_{j=1}^{N_M} dx_j^{(M)}$$
$$\times \det_{\substack{1\le i,j\le M \\ 1\le k\le N_i; 1\le l\le N_j}} \left[\mathcal{K}(t_i, x_k^{(i)}; t_j, x_l^{(j)})\right].$$

Theorem 2 thus follows from the pointwise convergence of the Hermite kernel to the Airy kernel in a proper rescaling and some additional bounds which ensure the convergence of the corresponding Fredholm determinants.

References

1. Weiß, T.: Scaling behaviour of the directed polymer model of Baryshnikov and O'Connell at zero temperature, *Bachelor's Thesis*, TU München (2011).
2. O'Connell, N. and Yor, M.: A Representation for non-colliding Random Walks, *Elect. Comm. in Probab.* 7, p. 1-12 (2002)
3. Praehofer, M. and Spohn, H.: Scale Invariance of the PNG Droplet and the Airy Process, *J. Stat. Phys.* 108, 5-6 p. 1071-1106 (2002)