A Variational Analysis of thin rigid inclusions in elastic bodies

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Abstract

The following is about the application of Γ-convergence to a problem coming from mathematical elasticity.

Elasticity Theory

As elasticity theory is a part of continuum mechanics the basic assumption is that all states of the bodies considered can be described by functions defined on a continuum. One may assume that those functions can be derived from atomistic models by averaging in a appropriate scale. In elasticity theory the object of interrest is a body occupying a domain $\Omega \subset \mathbb{R}^3$ while no forces are acting on it. The question one wants to answer then is how the object will be deformed when forces (surface forces, body forces, preassures...) are acting on it and what is the equilibrium that is reaches.

Given that the fundamental axiom of Euler and Cauchy and some smoothness assumptions hold, the reaction of a body to an applied body force with density f^φ and a surface-force with density g^φ can be described by

$$\begin{aligned} -\mathrm{div}^\varphi T^\varphi(x^\varphi) &= f^\varphi(x^\varphi) \qquad \forall x^\varphi \in \Omega^\varphi && (1)\\ T^\varphi(x^\varphi) &= T^\varphi(x^\varphi)^T \qquad x^\varphi \in \bar{\Omega}^\varphi && (2)\\ T^\varphi(x^\varphi)n^\varphi &= g^\varphi(x^\varphi) \qquad \forall x^\varphi \in \Gamma_1^\varphi && (3) \end{aligned}$$

where da-meass$\{\partial\Omega \setminus \Gamma_0^\varphi \cup \Gamma_1^\varphi\} = 0$ (in the deformed configuration).The Cauchy-stress-tensor $T^\varphi(x^\varphi)$ models the reaction of the material to the applied forces.
A material is counsidered elastic if

$$T^\varphi(x^\varphi) = \hat{T}^D(x, \nabla\varphi(x))$$

where $\hat{T}$ is the response function in the deformed configuration.
A material is called hyperelastic if the response function can be written as the Gâteaux derivative of a function $\hat{W} : \bar{\Omega} \times \mathbb{R}^{3,3} \to \mathbb{R}$ such that

$$\hat{T}^D(x, F) = \frac{\partial \hat{W}}{\partial F}(x, F)$$

If one further assumes that the applied forces are conservative , that is that there are functionals F and G sucht that:

$$F'(\varphi)\theta = \int\limits_\Omega f(x) \cdot \theta(x) dx \qquad G'(\varphi)\theta = \int\limits_{\Gamma_1} g(x) \cdot \theta(x) da$$

if one further assumes a hyperelastic material one can define the stored energy functional I:

$$I(\psi) = \int\limits_\Omega \hat{W}(x, \nabla\psi(x))dx - (F(\psi) + G(\psi))$$

and any smooth enough φ that solves $I(\varphi) = \inf\limits_{\psi\in\Phi} I(\psi)$ with $\varphi \in \Phi := \{\psi : \bar{\Omega} \to \mathbb{R}^3; \psi = \varphi_0 \text{ on } \Gamma_0\}$ also solves (1) and (3).

Γ-convergence

Γ-convergence is a tool that gives one convergence of (global!) minima of a sequence of functionals (F_h) to the (global) minimum of the Γ-limit functional under some coercivity assumption.
Definition: The Γ-lower limit and the Γ-upper limit of the sequence (F_h) are functions from X into $\bar{\mathbb{R}}$ defined by:

$$(\Gamma - \liminf_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y)$$

$$(\Gamma - \limsup_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y)$$

If there exists a function $F : X \to \bar{\mathbb{R}}$ such that $\Gamma - \liminf\limits_{h\to\infty} F_h = \Gamma - \limsup\limits_{h\to\infty} F_h = F$, then we write $F = \Gamma - \lim\limits_{h\to\infty} F_h$ and we say that the sequence (F_h) Γ-converges to F in X or that F is the Γ-limit of (F_h) (in X).

Definition: We say that the sequence (F_h) is equi-coercive (on X) if for every $t \in \mathbb{R}$ there exists a closed and countably compact subset K_t of X such that $\{F_h \leq t\} \subseteq K_t$ for every $h \in \mathbb{N}$.

Theorem: Suppose that (F_h) is equi-coercive in X. Then F' and F'' are coercive and

$$\min_{x\in X} F'(x) = \liminf_{h\to\infty} \inf_{x\in X} F_h(x)$$

If in addition (F_h) Γ-converges to a function F in X then F is coercive and

$$\min_{x\in X} F(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x)$$

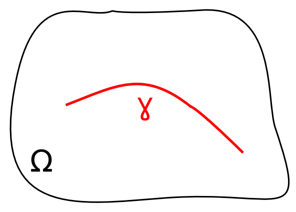
Not only this, there even holds a convergence-result for minimizers!
Theorem: Assume that (F_h) Γ-converges to a function F in X. For every $h \in \mathbb{N}$, let x_h be a minimizer of F_h in X. If x is a cluster point of (x_h) , then x is a minimizer of F in X and

$$F(x) = \limsup_{h\to\infty} F_h(x_h)$$

If (x_h) converges to x in X then x is a minimizer of F in X and

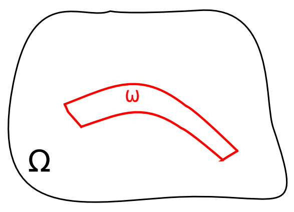
$$F(x) = \lim_{h\to\infty} F_h(x_h)$$

Application



$$\begin{aligned} \mathrm{div}(\sigma) &= f && \text{in } \Omega_\gamma\\ \sigma - Ae(u) &= 0 && \text{in } \Omega\\ u &= 0 && \text{on } \Gamma\\ u &= \rho_0 && \text{on } \gamma\\ \int_\gamma [\sigma n]\rho &= 0 && \forall\ \rho \text{ in } R(\gamma) \end{aligned}$$

The picture on the left-hand side shows a thin rigid inclusion in an elastic body. The rigidness of the inculusion is modelled by the restriction of the displacemnt u to the set
$R(U) := \{\rho \in, \mathbb{R}^2 \mid \rho(x) = Bx + c\ x \in U; B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}; c \in \mathbb{R}^2, b \in \mathbb{R}\ const.\}$
The picture on the right-hand side shows the same problem for the thick rigid inclusion. It is obvious that the system of differential equations of the thick rigid inclusion formally goes over to the one of the thin rigid inclusion when the thickness of the inclusion goes to zero.
This assumption can be rigorously verified by going over to a weak formulation of the PDE-systems that come out to be equivalent to minimization problems and then applying Γ-convergence methods.



$$\begin{aligned} \mathrm{div}(\sigma) &= f && \text{in } \Omega \setminus \omega\\ \sigma - Ae(u) &= 0 && \text{in } \Omega \setminus \omega\\ u &= 0 && \text{on } \Gamma\\ u &= \rho_0 && \text{on } \omega\\ \int_{\partial\omega} \sigma n \cdot \rho &= \textstyle\int_\omega f\rho && \forall\ \rho\ in\ R(\omega) \end{aligned}$$