

Montag, 3. November 2008

PETER HEINIG (TU MÜNCHEN)

Combinatorics meets Complex Analysis: From the Residue Theorem to concrete Counting Formulas

As we all know, the number of distinct words of length ℓ that can be formed from n distinct letters, no two of which commute, is exactly n^ℓ . If on the contrary any two of the n letters commute (an example for such a situation is the question “How many monomials of total degree ℓ are there in $K[x_1, \dots, x_n]$?”) then the number of words of length ℓ is exactly $\binom{n-1+\ell}{n-1}$.

For the obvious generalization to the case of partial commutativity rules there has been found a systematic solution twenty years ago, which — albeit very probably not efficient, let alone practical — for certain commutativity rules of manageable complexity yields explicit and surprisingly bizarre counting formulas. For example, if n is even and each letter has exactly one other letter with which it commutes (for instance let the letters be $x_1, X_1, \dots, x_{\frac{n}{2}}, X_{\frac{n}{2}}$ where x_i commutes with X_i and with no other) then the number of words of length ℓ is precisely

$$\frac{1}{2} \cdot \frac{1}{\sqrt{\frac{n^2}{4} - \frac{n}{2}}} \cdot \left(\left(\frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{n}{2}} \right)^{\ell+1} - \left(\frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{n}{2}} \right)^{\ell+1} \right).$$

Try to prove this by ad-hoc reasoning!

At one point, a very simple application of the residue theorem for holomorphic functions turns out to be helpful to get hold of concrete counting formulas. The residue theorem in this case owes its usefulness to a special property of a special graph polynomial.

I recently realized that I already knew a formula intriguingly similar (but not equal) to the one given above, but as yet I do not have any explanation for this similarity: The characteristic polynomial of the adjacency matrix of a path graph with ℓ vertices is

$$\frac{1}{2} \cdot \frac{1}{\sqrt{\frac{z^2}{4} - 1}} \cdot \left(\left(\frac{z}{2} + \sqrt{\frac{z^2}{4} - 1} \right)^{\ell+1} - \left(\frac{z}{2} - \sqrt{\frac{z^2}{4} - 1} \right)^{\ell+1} \right).$$

Do you have an explanation for this?